

# Bijjective proofs of Gould-Mohanty's and Raney-Mohanty's identities

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**Abstract.** Using the model of words, we give bijective proofs of Gould-Mohanty's and Raney-Mohanty's identities, which are respectively multivariable generalizations of Gould's identity

$$\sum_{k=0}^n \binom{x-kz}{k} \binom{y+kz}{n-k} = \sum_{k=0}^n \binom{x+\epsilon-kz}{k} \binom{y-\epsilon+kz}{n-k}$$

and Rothe's identity

$$\sum_{k=0}^n \frac{x}{x-kz} \binom{x-kz}{k} \binom{y+kz}{n-k} = \binom{x+y}{n}.$$

## 1. Introduction

A famous generalization of the binomial theorem is Abel's identity [1]:

$$\sum_{k=0}^n \binom{n}{k} x(x-kz)^{k-1} (y+kz)^{n-k} = (x+y)^n, \quad (1)$$

which also has a company identity as follows:

$$\sum_{k=0}^n \binom{n}{k} xy(x-kz)^{k-1} (y+kz)^{n-k-1} = (x+y+nz)(x+y)^{n-1}. \quad (2)$$

It is not difficult to see that (1) and (2) are respectively limiting cases of the following convolution formulas due to Rothe [17]:

$$\sum_{k=0}^n \frac{x}{x-kz} \binom{x-kz}{k} \binom{y+kz}{n-k} = \binom{x+y}{n}, \quad (3)$$

$$\begin{aligned} \sum_{k=0}^n \frac{xy}{(x-kz)(y-(n-k)z)} \binom{x-kz}{k} \binom{y-(n-k)z}{n-k} \\ = \frac{x+y}{x+y-nz} \binom{x+y-nz}{n}. \end{aligned} \quad (4)$$

Gould [5, 6] reproved (3) and (4) and also obtained the following identity

$$\sum_{k=0}^n \binom{x-kz}{k} \binom{y+kz}{n-k} = \sum_{k=0}^n \binom{x+\epsilon-kz}{k} \binom{y-\epsilon+kz}{n-k}. \quad (5)$$

Another proof of (3) and (4) was given by Sprugnoli [19]. It is not difficult to see that (4) can be deduced from (3). Blackwell and Dubins [2] gave a combinatorial proof of Rothe's identity (4), which can also be proved in the model of lattice paths (using [13, p. 9] or [10, (1.1)]). Recently, the author [8] gives simple bijective proofs of Gould's identity (5) and Rothe's identity (3) in the model of binary words.

Hurwitz [9] established a multivariable generalization of Abel's identities (1) and (2) (see also [20]). For a curious  $q$ -analogue of Rothe's identity (3), we refer the reader to [18] and references therein.

In order to state a multivariable generalization of Rothe's identities in the literature, we need first to introduce some notation. Let  $m$  be a fixed natural number throughout the paper. For  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$  and  $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{C}^m$ , set  $|\mathbf{a}| = a_1 + \dots + a_m$ ,  $\mathbf{a}! = a_1! \dots a_m!$ ,  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, \dots, a_m + b_m)$ ,  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \dots + a_m b_m$ , and  $\mathbf{b}^{\mathbf{a}} = b_1^{a_1} \dots b_m^{a_m}$ . For any complex parameter  $x$  and  $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$ , we define the *multinomial coefficient*  $\binom{x}{\mathbf{n}}$  by

$$\binom{x}{\mathbf{n}} = \begin{cases} x(x-1) \cdots (x - |\mathbf{n}| + 1) / \mathbf{n}!, & \text{if } \mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m, \\ 0, & \text{otherwise.} \end{cases}$$

Using generating functions, Mohanty [12] proved the following multivariable generalization of Rothe's identities (3) and (4):

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{x}{x - \mathbf{k} \cdot \mathbf{z}} \binom{x - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \binom{x + y}{\mathbf{n}}, \quad (6)$$

$$\begin{aligned} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{xy}{(x - \mathbf{k} \cdot \mathbf{z})(y - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z})} \binom{x - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \\ = \frac{x + y}{x + y - \mathbf{n} \cdot \mathbf{z}} \binom{x + y - \mathbf{n} \cdot \mathbf{z}}{\mathbf{n}}. \end{aligned} \quad (7)$$

However, an important special case of (7) (where  $z_i = i$ ) was already contained in the earlier work of Raney [16] on a combinatorial approach to the Lagrange inversion. Hence we would call both (6) and (7) *Raney-Mohanty's identities*. Unaware of Mohanty's work, in 1988 Louck [11] proposed a "conjecture" equivalent to (7), which caught the interests of three different people independently and was solved by them by three different methods: Paule [15] proved (7) by the Lagrange inversion approach, Strehl [20] gave a completely combinatorial approach, while Zeng [21] used mathematical induction.

Moreover, Mohanty and Handa [14] established the following identity

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{x + y - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \quad (8)$$

which is a multivariable generalization of Jensen's identity [7]:

$$\sum_{k=0}^n \binom{x + kz}{k} \binom{y - kz}{n - k} = \sum_{k=0}^n \binom{x + y - k}{n - k} z^k.$$

It follows immediately from Mohanty-Handa's identity (8) that

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{x - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{x + \epsilon - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \epsilon + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}}. \quad (9)$$

Since (9) is obviously a multivariable generalization of Gould's identity (5) and it also follows from one of the generating functions established by Mohanty in [12], we call (9) *Gould-Mohanty's identity*.

To the knowledge of the author, there are no combinatorial proofs of Mohanty-Handa's identity (8) and Gould-Mohanty's identity (9). In this paper, continuing the work of [8], we shall give bijective proofs of Gould-Mohanty's identity and Raney-Mohanty's identity (6) in the model of words.

## 2. Proof of Gould-Mohanty's identity

It suffices to prove Gould-Mohanty's identity (9) for the special case:

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{q + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p + 1 - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{q - 1 + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}}, \quad (10)$$

where  $p, q \in \mathbb{N}$  and  $\mathbf{n}, \mathbf{z} \in \mathbb{N}^m$ . Furthermore, we need only to prove that (10) holds for all integers  $p \geq \mathbf{n} \cdot \mathbf{z}$  and  $q \geq 1$ . In this case, each multinomial coefficient in (10) is nonnegative and therefore has a combinatorial interpretation.

Let  $\Gamma = \{a, b_1, \dots, b_m\}$  denote an alphabet with a grading  $\|a\| = 1$  and  $\|b_i\| = z_i + 1$  ( $1 \leq i \leq m$ ). For a word  $w = w_1 \cdots w_n \in \Gamma^*$ , its *length*  $n$  is denoted by  $|w|$  and its *weight* by  $\|w\| = \|w_1\| + \cdots + \|w_n\|$ , and we call the word  $w_n w_{n-1} \cdots w_1$  the *reverse* of  $w$ . Let  $|w|_{b_i}$  be the number of  $b_i$ 's appearing in  $w$ , and let

$$\Gamma_{p, \mathbf{k}} := \{w \in \Gamma^* : \|w\| = p \text{ and } |w|_{b_i} = k_i, i = 1, \dots, m\},$$

where  $\mathbf{k} = (k_1, \dots, k_m)$ . It is easy to see that  $\Gamma_{p, \mathbf{k}} \subseteq \Gamma^{p - \mathbf{k} \cdot \mathbf{z}}$  and

$$\#\Gamma_{p, \mathbf{k}} = \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}}, \quad (11)$$

where  $\mathbf{z} = (z_1, \dots, z_m)$ .

Furthermore, let

$$\Gamma_{p, \mathbf{k}}^{(r)} := \{w \in \Gamma_{p, \mathbf{k}} : w \text{ has a prefix of weight } r\}.$$

For  $p, q \geq \mathbf{n} \cdot \mathbf{z}$ , an obvious bijection

$$\Gamma_{p+q, \mathbf{n}}^{(p)} \longleftrightarrow \bigsqcup_{\mathbf{k}} \Gamma_{p, \mathbf{k}} \times \Gamma_{q, \mathbf{n} - \mathbf{k}}$$

leads to

$$\#\Gamma_{p+q, \mathbf{n}}^{(p)} = \sum_{\mathbf{k}} \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{q - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}}. \quad (12)$$

Thus, the identity (10) is equivalent to

$$\#\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)} = \#\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p+1)}. \quad (13)$$

We need the following simple fact.

**Lemma 1.** *Let  $u, v \in \Gamma^*$  with  $\|u\|, \|v\| \geq \mathbf{n} \cdot \mathbf{z} + 1$ , where  $n_i = |u \cdot v|_{b_i}$  ( $1 \leq i \leq m$ ). Then there exist nonempty prefixes  $x$  of  $u$  and  $y$  of  $v$  such that  $\|x\| = \|y\|$ .*

*Proof.* Since the proof is easy and very similar to the proof of [8, Lemma 1], we omit it here.  $\square$

Now we can prove (13) by the following theorem.

**Theorem 2.** *For all  $p \geq \mathbf{n} \cdot \mathbf{z}$  and  $q \geq 1$ , there is a bijection between  $\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)}$  and  $\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p+1)}$ .*

*Proof.* Suppose that  $w = u \cdot v \in \Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)}$ , where  $\|u\| = p$  and  $\|v\| = q + \mathbf{n} \cdot \mathbf{z}$ . Applying Lemma 1 to  $v$  and the reverse of  $u \cdot a$ , one sees that  $u$  has a suffix  $x$  (perhaps empty), i.e.,  $u = u' \cdot x$ , and  $v$  has a prefix  $y$ , i.e.,  $v = y \cdot v'$ , such that  $\|x\| = \|y\| - 1$ . Choosing such  $x$  and  $y$  with minimal length, then  $w' = u' \cdot \bar{y} \cdot \bar{x} \cdot v' \in \Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p+1)}$  and  $w \mapsto w'$  is a bijection. Here  $\bar{x}$  and  $\bar{y}$  are respectively the reverses of  $x$  and  $y$ .  $\square$

In the same manner, we may also give a direct bijection from  $\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)}$  to  $\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p+r)}$  for all  $p \geq \mathbf{n} \cdot \mathbf{z}$  and  $q \geq r \geq 1$ .

### 3. Proof of Raney-Mohanty's identity

We again assume that  $p \geq \mathbf{n} \cdot \mathbf{z}$  and  $q \geq 1$ . Moreover, let  $z_i \geq 1$  for all  $i$ . For each  $w \in \Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}$ , let  $w = u \cdot v$  denote the unique factorization with  $\|u\| \geq p$  but as small as possible. Then we have the following possibilities:

- If  $\|u\| = p$ , then  $w \in \Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)}$  and all these words have been counted in Section 2.
- If  $\|u\| = p + j$  for some  $1 \leq j \leq \max\{z_1, \dots, z_m\}$ , then the last letter of  $u$  must be  $b_i$  for some  $1 \leq i \leq m$ . Namely,  $u = u' \cdot b_i$  for some  $u' \in \Gamma_{p+j-z_i-1, \mathbf{k}-\mathbf{e}_i}$ , where  $\mathbf{e}_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^m$  with the 1 being in the  $i$ -th position. The corresponding  $v$  belongs to  $\Gamma_{q+\mathbf{n} \cdot \mathbf{z}-j, \mathbf{n}-\mathbf{k}}$ . It is clear that the mapping  $w \mapsto (u', v)$  may be inverted.

Hence there is a bijection

$$\Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}} \longleftrightarrow \Gamma_{p+q+\mathbf{n} \cdot \mathbf{z}, \mathbf{n}}^{(p)} \bigcup_{i=1}^m \bigcup_{j=1}^{z_i} \bigcup_{\mathbf{k}=0}^{\mathbf{n}} \Gamma_{p+j-z_i-1, \mathbf{k}-\mathbf{e}_i} \times \Gamma_{q+\mathbf{n} \cdot \mathbf{z}-j, \mathbf{n}-\mathbf{k}},$$

which, together with (11) and (12), gives the identity

$$\begin{aligned} \sum_{\mathbf{k}=0}^{\mathbf{n}} \left( \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{q + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \right. \\ \left. + \sum_{i=1}^m \sum_{j=1}^{z_i} \binom{p - \mathbf{k} \cdot \mathbf{z} + j - 1}{\mathbf{k} - \mathbf{e}_i} \binom{q + \mathbf{k} \cdot \mathbf{z} - j}{\mathbf{n} - \mathbf{k}} \right) = \binom{p + q}{\mathbf{n}}. \end{aligned} \quad (14)$$

However, by (9), for all  $1 \leq i \leq m$  and  $1 \leq j \leq z_i$ , we have

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p - \mathbf{k} \cdot \mathbf{z} + j - 1}{\mathbf{k} - \mathbf{e}_i} \binom{q + \mathbf{k} \cdot \mathbf{z} - j}{\mathbf{n} - \mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p - \mathbf{k} \cdot \mathbf{z} - 1}{\mathbf{k} - \mathbf{e}_i} \binom{q + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}}. \quad (15)$$

Substituting (15) into (14), we obtain

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \left( \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} + \sum_{i=1}^m z_i \binom{p - \mathbf{k} \cdot \mathbf{z} - 1}{\mathbf{k} - \mathbf{e}_i} \right) \binom{q + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \binom{p + q}{\mathbf{n}}. \quad (16)$$

Noticing that

$$\binom{p - \mathbf{k} \cdot \mathbf{z} - 1}{\mathbf{k} - \mathbf{e}_i} = \frac{k_i}{p - \mathbf{k} \cdot \mathbf{z}} \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}},$$

the identity (16) may be simplified as

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{p}{p - \mathbf{k} \cdot \mathbf{z}} \binom{p - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{q + \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} = \binom{p + q}{\mathbf{n}},$$

which is Raney-Mohanty's identity (6).

For the  $m = 1$  case, the above bijection also leads to a double sum extension of the  $q$ -Chu-Vandermonde formula (see [8]). It is also possible to give a similar  $q$ -analogue of (14). However we omit it here and leave it to the interested reader.

## 4. Some remarks

We point out that (7) is a consequence of (6), since the left-hand side of the former may be written as

$$\begin{aligned} \frac{1}{x + y - \mathbf{n} \cdot \mathbf{z}} \left( \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{xy}{x - \mathbf{k} \cdot \mathbf{z}} \binom{x - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \right) \\ + \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{xy}{y - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z}} \binom{x - \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - (\mathbf{n} - \mathbf{k}) \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \right). \end{aligned}$$

It is also worth mentioning that Mohanty-Handa's identity (8) can be deduced from Raney-Mohanty's identity (6). Indeed, note that

$$\begin{aligned} \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} &= \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{x}{x + \mathbf{k} \cdot \mathbf{z}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \\ &\quad + \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{\mathbf{k} \cdot \mathbf{z}}{x + \mathbf{k} \cdot \mathbf{z}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}} \\ &= \binom{x + y}{\mathbf{n}} + \sum_{i=1}^m \sum_{\mathbf{k}=0}^{\mathbf{n}} z_i \binom{x - 1 + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k} - \mathbf{e}_i} \binom{y - \mathbf{k} \cdot \mathbf{z}}{\mathbf{n} - \mathbf{k}}. \end{aligned}$$

Then (8) follows from (6) by induction on  $|\mathbf{n}|$ . However, I am unable to give a combinatorial proof of Mohanty-Handa's identity.

Finally, we remark that a further generalization of (8) was given by Chu [3] by using the following generating functions due to Mohanty [12]:

$$\begin{aligned} \sum_{\mathbf{k} \geq 0} \frac{x}{x + \mathbf{k} \cdot \mathbf{z}} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} u_1^{k_1} \cdots u_m^{k_m} &= v^x, \\ \sum_{\mathbf{k} \geq 0} \binom{x + \mathbf{k} \cdot \mathbf{z}}{\mathbf{k}} u_1^{k_1} \cdots u_m^{k_m} &= \frac{v^x}{1 - \sum_{i=1}^m u_i z_i v^{z_i - 1}}, \end{aligned}$$

where  $v$  satisfies the functional equation  $\sum_{i=1}^m u_i v^{z_i} = v - 1$ .

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